The expected sum of edge lengths in planar linearizations of trees

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ABSTRACT

Dependency trees have proven to be a very successful model to represent the syntactic structure of sentences of human languages. In these structures, vertices are words and edges connect syntactically-dependent words. The tendency of these dependencies to be short has been demonstrated using random baselines for the sum of the lengths of the edges or their variants. A ubiquitous baseline is the expected sum in projective orderings (wherein edges do not cross and the root word of the sentence is not covered by any edge), that can be computed in time $O(n)$. Here we focus on a weaker formal constraint, namely planarity. In the theoretical domain, we present a characterization of planarity that, given a sentence, yields either the number of planar permutations or an efficient algorithm to generate uniformly random planar permutations of the words. We also show the relationship between the expected sum in planar arrangements and the expected sum in projective arrangements. In the domain of applications, we derive a $O(n)$-time algorithm to calculate the expected value of the sum of edge lengths. We also apply this research to a parallel corpus and find that the gap between actual dependency distance and the random baseline reduces as the strength of the formal constraint on dependency structures increases, suggesting that formal constraints absorb part of the dependency distance minimization effect. Our research paves the way for replicating past research on dependency distance minimization using random planar linearizations as random baseline.

Keywords:
dependency grammar, projectivity, planarity, syntactic dependency distance minimization

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The structure of a natural language sentence can be represented as a (labelled) graph indicating the syntactic relationships between words together with the encoding of the words' order. In such a graph, the edge labels indicate the type of syntactic relationship between the words. Such a combination of graph and linear ordering, as in Figure 1, is known as syntactic dependency structure (Nivre 2006). When the graph is (1) well-formed, namely, the graph is weakly connected, (2) is acyclic, that is, there are no cycles in the graph, (3) is single-headed, that is, every node has a single head (except for the root node), and (4) there is only one root node (one node with no head) in the graph, then it is called a syntactic dependency tree (Nivre 2006). There exist formal constraints that are often imposed on dependency structures. One such constraint is projectivity: a dependency structure is projective if, for every vertex $v$, all vertices reachable from $v$ in the underlying graph form a continuous substring within the sentence (Kuhlmann and Nivre 2006). Projectivity implies that (1) the root word of the sentence (the root of the underlying syntactic dependency structure) is never covered (as in Figure 1(a)) and (2) planarity, namely absence of edge crossings (Figure 1(a) and (b)). Indeed planarity is another constraint that generalizes projectivity by allowing the root to be covered by one or more edges (as in Figure 1(b)). Figure 1(c) shows a sentence that is neither projective nor planar.

In this article, we study statistical properties of syntactic dependency structures under the planarity constraint. Such structures are represented in this article as a pair consisting of a (free or rooted) tree and a linear arrangement of its vertices. Free trees are denoted as $T = (V,E)$, and rooted trees as $T^r = (V,E;r)$, where $V$ is the set of vertices, $E$ the set of edges, and $r \in V$ denotes the root vertex. Unless stated otherwise $n = |V|$, that is, $n$ denotes the number of vertices which is equal to the number of words in the sentence. A linear arrangement $\pi$ (also called embedding) of a tree is a (bijective) function ($\pi : V \to \{1,\ldots,n\}$) that maps every vertex $u$ of a tree to a unique position in $\{1,\ldots,n\}$, which is denoted by $\pi(u)$.

Projectivity, as well as planarity, can be alternatively defined on linear arrangements using the concept of edge crossing. We say that
any two (undirected) edges \{s, t\}, \{u, v\} cross if the positions of their vertices interleave. More formally, assume, without loss of generality, that \(\pi(s) < \pi(t), \pi(u) < \pi(v)\) and \(\pi(s) < \pi(u)\). Then, edges \{s, t\}, \{u, v\} cross in the linear ordering defined by \(\pi\) if \(\pi(s) < \pi(u) < \pi(t) < \pi(v)\).\(^1\) We denote the total number of edge crossings in an arrangement \(\pi\) as \(C_\pi(T)\). Then, an arrangement \(\pi\) of a rooted tree \(T^r\) is planar if \(C_\pi(T^r) = 0\) and is projective if (a) it is planar and (b) the root of the tree is not covered, that is, there is no edge \{s, t\} such that \(\pi(s) < \pi(r) < \pi(t)\) or \(\pi(t) < \pi(r) < \pi(s)\). Planarity is a relaxation of projectivity where the root can be covered (Sleator and Temperley 1993; Kuhlmann and Nivre 2006). Planar arrangements are also known in the literature as one-page book embeddings (Bernhart and Kainen 1979).

In this article, the main object of study is the expectation of the sum of edge lengths (or syntactic dependency distances) in planar arrangements of free trees. The length of an edge connecting two syntactically-related words, also known as dependency distance, is usually\(^2\) defined as the number of intervening words between \(u\) and \(v\)

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\(^1\) Notice that this notion of crossing does not depend on edge orientation.  
\(^2\) Another popular definition is \(\delta_{u,v}(\pi) = |\pi(u) - \pi(v)| - 1\) (Liu et al. 2017).
Figure 2: Examples of sentences with their syntactic dependency structures; arc labels indicate dependency distance. The rectangles denote the root word in each sentence. Examples adapted from Morrill 2000. The sum of edge lengths are $D = 18$ for (a) and $D = 12$ for (b) in the sentence plus 1 (Figure 1). It is defined mathematically as

$$\delta_{uv}(\pi) = |\pi(u) - \pi(v)|.$$  

We define the total sum of edge lengths in $\pi$ as

$$D_\pi(T) = \sum_{uv \in E} \delta_{uv}(\pi).$$

Close attention has been paid to this metric in modern linguistic research since its causal relationship with cognitive cost was first put forward, to the best of our knowledge, by Hudson 1995. The main causal argument is that the longer the dependency, the greater the memory burden arising from decay of activation and interference (Hudson 1995; Liu et al. 2017). A number of studies have exposed the general tendency in languages to reduce $D$, the total sum of edge lengths, a reflection of a potentially universal cognitive force known as the Dependency Distance Minimization principle (DDm) (Ferrer-i-Cancho 2004; Liu 2008; Futrell et al. 2015; Liu et al. 2017; Ferrer-i-Cancho et al. 2022). As an example of such cognitive cost, consider the sentences in Figures 2(a) and 2(b): it is not surprising that the latter is preferred over the former due to smaller total sum of edge lengths (Morrill 2000), the former's being $D = 18$ and the latter's being $D = 12$.

Statistical evidence of the DDm principle has been provided showing that dependency distances are smaller than expected by chance in syntactic dependency treebanks (Ferrer-i-Cancho 2004; Liu 2008; Park and Levy 2009; Gildea and Temperley 2010; Futrell et al. 2015; Liu...
et al. 2017; Ferrer-i-Cancho et al. 2022; Kramer 2021). Typically, the random baseline is defined as a random shuffling of the words of a sentence. To the best of our knowledge, the first known instance of such an approach was done by Ferrer-i-Cancho 2004, who established the DDm principle by comparing the average real $D(T)$ of sentences against the corresponding expected value in a uniformly random permutation of sentences’ words. More formally, Ferrer-i-Cancho 2004 calculated the expected value of $D(T)$ when the words of the sentence are shuffled uniformly at random (u.a.r.), that is, when all $n!$ permutations are equally likely. This value is denoted here as $\mathbb{E}[D(T)]$. Ferrer-i-Cancho 2004 found that

\begin{equation}
\mathbb{E}[D(T)] = \frac{n^2 - 1}{3}.
\end{equation}

In spite of the simplicity of Equation 2, the majority of researchers have used as random baseline the expected sum of edge lengths conditioned to projective arrangements (Temperley 2008; Park and Levy 2009; Gildea and Temperley 2010; Futrell et al. 2015; Kramer 2021) which we denote here as $\mathbb{E}_{pr}[D(T')]$. However, this baseline has been computed approximately via random sampling of projective arrangements. For these reasons, a formula to calculate the exact value of $\mathbb{E}_{pr}[D(T')]$ in linear time was derived by Alemany-Puig and Ferrer-i-Cancho 2022

\begin{equation}
\mathbb{E}_{pr}[D(T')] = \frac{1}{6} \sum_{u \in V} s_r(u)(2d_r(u) + 1) - \frac{1}{6},
\end{equation}

where $s_r(u)$ denotes the size (in vertices) of the subtree of $T'$ rooted at $u$, and $d_r(u)$ is the out-degree of $u$ in $T'$. In spite of its extensive use, the projective random baseline has some limitations. First, the percentage of non-projective sentences in languages ranges between 18.2 and 26.4 (Gómez-Rodríguez 2016) or between 6.8 and 36.4 (Gómez-Rodríguez and g 2010) (see also Havelka 2007). The limited coverage of projectivity raises the question if the projective baseline should be used for sentences that are not projective as it is customary in research on dependency distance minimization. In addition, projectivity per se implies a reduction in dependency distances, which raises the question if that rather strong constraint may mask the effect of the dependency distance minimization principle under investigation (Gómez-
Rodríguez et al. 2022). Here we aim to make a step forward by considering planarity, a generalization of projectivity, so as to increase the coverage of real sentences and reduce the bias towards dependency minimization in the random baseline. The percentage of non-planar sentences in languages ranges between 14.3 and 20.0 (Ferrer-i-Cancho et al. 2018) or between 5.3 and 31 (Gómez-Rodríguez and 2010). The latter range is consistent with earlier estimates (Havelka 2007).

This article is part of a research program on the statistical properties of $D(T)$ under constraints on the possible linear arrangements (Ferrer-i-Cancho 2019; Alemany-Puig et al. 2022; Alemany-Puig and Ferrer-i-Cancho 2022). The remainder of the article is divided into two main parts: theory (Section 2) and applications (Section 3).

The theory part (Section 2) is structured as follows. In Section 2.1, we introduce notation used throughout that part. In Section 2.2, we first present a characterization of planar arrangements so as to identify their underlying structure, which we apply to count their number for a given free tree, and later on in Section 2.3, to generate them u.a.r. by means of a novel $O(n)$-time algorithm. In Section 2.4, we use said characterization to prove the main result of the article, namely that expectation of $D(T)$ in planar arrangements can be calculated from the expectation of projective arrangements, as the following theorem indicates.

**THEOREM 1** Given a free tree $T = (V, E)$,

$$E_{pl}[D(T)] = \frac{1}{n} \sum_{u \in V} E_{pr}^\circ[D(T_u^u)]$$

$$= \frac{(n-1)(n-2)}{6n} + \frac{1}{n} \sum_{u \in V} E_{pr}[D(T_u^u)],$$

where $E_{pr}^\circ[D(T_u^u)]$ is the expected value of $D(T_u^u)$ in uniformly random projective arrangements $\pi$ of $T_u^u$ such that $\pi(u) = 1$ and $E_{pr}[D(T_u^u)]$ (Equation 3) is the expected value of $D(T_u^u)$ in uniformly random projective arrangements of $T_u^u$, the free tree $T$ rooted at $u$.

Table 1 summarizes the theoretical results obtained in previous articles and those presented in this article.

The applications part (Section 3) is structured as follows. In Section 3.1, we apply Theorem 1 to derive a $O(n)$-time algorithm to calculate $E_{pl}[D(T)]$. Since Alemany-Puig and Ferrer-i-Cancho 2022
Table 1: Summary of the main mathematical results for increasing constraints on linear orders. Results for the unconstrained and projective cases are borrowed from previous research (Ferrer-i-Cancho 2004 and Alemany-Puig and Ferrer-i-Cancho 2022, respectively). Results for the planar case are a contribution of this article. $N_{pr}(T')$, $N_{pl}(T)$ and $N(T)$ denote the number of distinct projective, planar and unconstrained linear arrangements, respectively, of a rooted tree $T'$ or of a free tree $T$. $E_{pr}[\delta_{uv}]$, $E_{pl}[\delta_{uv}]$ and $E[\delta_{uv}]$ denote the expected length of an edge in random linear arrangement for the projective, planar and unconstrained cases, respectively. $E_{pr}[\delta_{uv} \mid s]$ is the expected value of $\delta_{uv}$ conditioned to having vertex $s$ as root of the tree. In $E_{pr}[\delta_{uv}]$ the root is vertex $r$.

<table>
<thead>
<tr>
<th></th>
<th>$N(T)$</th>
<th>$\mathbb{E}[\delta_{uv}]$</th>
<th>$\mathbb{E}[D(T)]$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Unconstrained ($T$)</strong></td>
<td>$n!$</td>
<td>$\frac{n+1}{3}$</td>
<td>$\frac{n^2-1}{3}$</td>
</tr>
<tr>
<td><strong>Planar ($T$)</strong></td>
<td>$N_{pl}(T)$</td>
<td>$\prod_{u \in V} d(u)!$</td>
<td>$1 + \frac{1}{n} \sum_{s \in \mathcal{V} \setminus {u,v}} E_{pr}[\delta_{uv} \mid s]$</td>
</tr>
<tr>
<td><strong>Projective ($T'$)</strong></td>
<td>$N_{pr}(T')$</td>
<td>$\prod_{u \in \mathcal{V}} (d_{r}(u) + 1)!$</td>
<td>$\frac{1}{6} (2s_{r}(u) + s_{r}(v) + 1)$</td>
</tr>
</tbody>
</table>

showed that $E_{pr}[D(T')]$ can be evaluated in time $O(n)$, Equation 5 naturally leads to a $O(n^2)$-time algorithm if it is evaluated ‘as is’. However, we devise a $O(n)$-time algorithm to calculate $E_{pl}[D(T')]$. In Section 3.2, we apply this and previous research on the projective case (Alemany-Puig and Ferrer-i-Cancho 2022) to a parallel syntactic dependency treebank. We find that the gap between the actual dependency distance and that of the random baseline reduces as the strength of the formal constraint on dependency structures chosen for the ran-
dom baseline increases, suggesting that formal constraints absorb part of the dependency distance minimization effect.

Finally, in Section 4, we review all the findings and make suggestions for future research.

From this point onwards, the article is organized to ease reading by readers of distinct profiles. Readers interested in the analysis of syntactic dependency treebanks can jump directly to Section 3.2. Readers interested in the algorithm for computing $E_{pl}[D(T)]$ can jump directly to Section 3.1, after reading Section 2.1. Readers whose primary interest is applying the algorithms have ready-to-use code: both methods to generate planar arrangements (Section 2.3) and the $O(n)$-time calculation of $E_{pl}[D(T)]$ (Section 3.1) are freely available in the Linear Arrangement Library\(^3\) (Alemany-Puig et al. 2021).

2 THEORY

2.1 Definitions and notation

We use $u, v, w, z$ to denote vertices, $r$ to always denote a root vertex, and $i, j, k, p, q$ to denote integers. The edges of a free tree are undirected, and denoted as $\{u, v\} = uv$; those of rooted trees are directed, denoted as $(u, v)$, and oriented away from $r$ towards the leaves.

Let $\Gamma(u)$ denote the set of neighbors of $u \in V$ in the free tree $T$, and let $\Gamma_r(u)$ denote the out neighbors (also, children) of $u \in V$ in $T_r$. Notice that, $\Gamma_r(u) \subseteq \Gamma(u)$ with equality if, and only if $u = r$. Let $d_r(u) = |\Gamma_r(u)|$ denote the out-degree of vertex $u$ of a rooted tree $T_r$, and let $d(u) = |\Gamma(u)|$ denote the degree of $u$ in a free tree $T$. Notice that $d_r(u) = d(u) - 1$ when $u \neq r$ and $d_r(r) = d(r)$. Furthermore, we denote the subtree rooted at $v$ with respect to root $u$ as $T^u_v$ (obviously $T^r_r = T^r$), and its size as $s_u(v) = |V(T^u_v)|$ (Figure 3). We call this directional size (Hochberg and Stallmann 2003; Alemany-Puig et al. 2022). Note that $s_v(u) + s_u(v) = n$ for any $uv \in E$.

\(^3\)https://github.com/LAL-project/linear-arrangement-library/
As in previous research, we also decompose an edge \((r, u)\) in a projective arrangement \(\pi\) into two parts: its anchor and its coanchor, as in Figure 4 (Shiloach 1979; Chung 1984; Alemany-Puig and Ferrer-i-Cancho 2022). Informally, \(\alpha_{ru}(\pi)\) is the number of vertices in \(\pi\) covered by \((r, u)\) in the segment of \(T_r^u\) including vertex \(u\) (Figure 4); similarly, \(\beta_{ru}(\pi)\), is the number of vertices of \(\pi\) covered by \((r, u)\) in segments that fall between \(r\) and \(u\) (Figure 4). The length of an edge connecting \(r\) with \(u\) can be expressed with the formula

\[
\delta_{ru}(\pi) = |\pi(r) - \pi(u)| = \alpha_{ru}(\pi) + \beta_{ru}(\pi),
\]

where \(\alpha_{ru}(\pi)\) is the length of the anchor and \(\beta_{ru}(\pi)\) is the length of the coanchor. The length of the anchor and coanchor can be formally defined as

\[
\begin{align*}
\alpha_{ru}(\pi) &= |\pi(u) - \pi(z)| + 1 \\
\beta_{ru}(\pi) &= |\pi(z) - \pi(r)| - 1,
\end{align*}
\]

where \(z \in V(T_r^u)\) is the vertex of \(T_r^u\) closest to \(r\) in \(\pi\) (Figure 4). The same notation with \(\pi\) omitted, \(\alpha_{ru}\) and \(\beta_{ru}\) denote random variables. Furthermore, it will be useful to define the operator \(\diamond\), which we use to condition expected values and constrain sets of arrangements of a rooted tree, in both cases to arrangements \(\pi\) where (only) the root is fixed at the leftmost position of \(\pi\). For instance, if \(S\) is a set of arrangements \(\pi\) of a rooted tree \(T^r\) then \(S^{\diamond} = \{\pi \in S \mid \pi(r) = 1\}\). Moreover, if \(X\) is defined on uniformly random arrangements from \(S\) then \(E^{\diamond}[X]\) is the expected value of \(X\) in uniformly random arrangements from \(S^{\diamond}\).

Finally, in this article we consider that two arrangements \(\pi\) and \(\pi'\) of the same tree \(T\) are different if there is (at least) one vertex \(u\) for which \(\pi(u) \neq \pi'(u)\).
2.2 Counting planar arrangements

It is well known that the number of unconstrained arrangements of an $n$-vertex tree is $n!$. This is true given that arrangements are simply permutations, and unconstrained arrangements are not subject to any particular constraint, thus all vertex orderings are possible. Building on the fact that projective arrangements span over contiguous intervals (Kuhlmann and Nivre 2006), Alemany-Puig and Ferrer-i-Cancho 2022 studied the expected value of the random variable $D(T^r)$ in such arrangements by defining, as usual, a set of segments $\Phi_u$ associated to each vertex $u$, consisting of the segments associated to the subtrees $T^r_{u_1}, \ldots, T^r_{u_p}$ and $u$. A segment of a rooted tree $T^r_u$ is a segment within the linear ordering containing all vertices of $T^r_u$, an interval of length $s_r(u)$ whose starting and ending positions are unknown until the whole tree is fully linearized; thus, a segment is a movable set of vertices within the linear ordering (Alemany-Puig and Ferrer-i-Cancho 2022). For a vertex $u$, the set $\Phi_u$ is constructed from vertex $u$’s segment and the segments of its children $\Gamma_r(u) = \{u_1, \ldots, u_k\}$ (Figure 5). Decomposing every vertex and its segments from the root to the leaves linearizes $T^r$ into a projective arrangement (Figure 5). This characterization led to a straightforward derivation of the number of projective arrangements of a rooted tree $T^r$ (Table 1)

$$N_{pr}(T^r) = \prod_{u \in V}(d_r(u) + 1)!.$$  

Using the structure of segments summarized above, we present a characterization of planar arrangements of free trees which helps to devise a method to generate planar arrangements u.a.r. (Section 2.3.3) and to prove Theorem 1 (Section 2.4). To this aim, we define $P^\circ_{pr}(T^r)$ as the set of projective arrangements of a rooted tree $T^r$ such that $\pi(r) = 1$, and denote its size as $N^\circ_{pr}(T^r) = |P^\circ_{pr}(T^r)|$. Notice that
when a vertex \( u \) is fixed to the leftmost position, the planar arrangements in \( P_{pr}^\circ(T^u) \) are obtained by arranging the subtrees \( T^v_\Gamma(v) \), \( v \in \Gamma(u) \), projectively to the right of \( u \) in the linear arrangement. It is important to bear in mind that the operator \( \circ \) only fixes the root vertex \( r \) to the leftmost position of the arrangement: the other vertices can be placed freely as long as the result is projective.

**Proposition 1**  
The number of planar arrangements of an \( n \)-vertex free tree \( T = (V,E) \), with \( V = \{u_1, \ldots, u_n\} \) is

\[
N_{pl}(T) = n N_{pr}^\circ(T^{u_1}) = \cdots = n N_{pr}^\circ(T^{u_n}) = n \prod_{u \in V} d(u)!.
\]

**Proof**  
Given a free tree \( T \), and any two distinct vertices \( u, v \), it holds that \( P_{pr}^\circ(T^u) \cap P_{pr}^\circ(T^v) = \emptyset \) because the vertices in the first positions are different. This lets us partition \( P_{pl}(T) \) into the non-empty
pairwise-disjoint sets $P^o_u(T^u)$ and see that

$$N_{pl}(T) = \sum_{u \in V} N_{pr}^o(T^u).$$

It is easy to see that

$$N^o_{pr}(T^u) = d(u)! \prod_{v \in \Gamma(u)} N_{pr}(T^u_v) = \prod_{v \in V} d(v)!.$$  

We used Equation 6 in the second equality. Notice that

$$N^o_{pr}(T^u_1) = \cdots = N^o_{pr}(T^u_n),$$

since the value $N^o_{pr}(T^u)$ does not depend on the root vertex $u$. Therefore, Equation 7 follows immediately. \qed

Obviously, there are more planar arrangements of a free tree $T$ than projective arrangements of any ‘rooting’ $T^r$ of $T$, formally $N_{pl}(T) \geq N_{pr}(T^r)$. We can see this by noticing that, when given a ‘rooting’ of $T$ at $r \in V$,

$$\frac{N_{pl}(T)}{N_{pr}(T^r)} = \frac{nd(r)! \prod_{u \in V \setminus \{r\}} d(u)!}{(d(r) + 1)! \prod_{u \in V \setminus \{r\}} d(u)!} = \frac{n}{d(r) + 1} \geq 1,$$

with equality when $T$ is a star tree\footnote{An $n$-vertex star tree consists of a vertex connected to $n - 1$ leaves; it is also a complete bipartite graph $K_{1,n-1}$.} and $r$ is its vertex of highest degree.

### 2.3 Generating arrangements uniformly at random

Arrangements can be generated freely, that is, by imposing no constraint on the possible orderings, where all the $n!$ possible orderings are equally likely, or by imposing some constraint on the possible orderings. Generating unconstrained arrangements is straightforward: it is well known that a permutation of $n$ elements can be generated u.a.r. in time $O(n)$ (Cormen et al. 2001). It can be done as follows. Assume we are given a set of $n$ vertices, say $V = \{u_1, \ldots, u_n\}$, and let $i = 1$. Repeat the following steps $n$ times:
1. select u.a.r. a vertex from $V$; the vertex is chosen with probability $1/(n - i + 1)$. Let $u_i$ be said vertex,
2. place $u_i$ in the arrangement at position $i$, that is, let $\pi(u_i) = i$,
3. remove $u_i$ from $V$,
4. increment $i$ by 1.

The product of all probabilities of vertex choice gives that the probability of producing a certain linear arrangement is

$$\prod_{i=1}^{n} \frac{1}{n - i + 1} = \frac{1}{n!}$$

thus the arrangement is constructed uniformly at random. Since the removal of a vertex from the set and uniformly random choice of vertex can both be implemented in constant time (using arrays), the running time is $O(n)$.

When constraints are involved, projectivity is often the preferred choice (Gildea and Temperley 2007; Liu 2008; Futrell et al. 2015). First, we present a $O(n)$-time procedure to generate projective arrangements u.a.r. (Section 2.3.1) and review methods used in past research (Section 2.3.2). Then we present a novel $O(n)$-time procedure to generate planar arrangements u.a.r. (Section 2.3.3) which in turn involves the generation of random projective arrangements of a subtree.

### 2.3.1 Generating projective arrangements

The method we will present in detail here was outlined first by Futrell et al. 2015. Here we borrow from recent theoretical research summarized above (Alemany-Puig and Ferrer-i-Cancho 2022) to derive a detailed algorithm to generate projective arrangements and prove its correctness. In order to generate projective arrangements u.a.r., simply make random permutations of a vertex $u$ and its children $\Gamma_r(u)$, that is, choose one of the possible $(d_r(u) + 1)!$ permutations u.a.r. Algorithm 1 formalizes this brief description. The proof that Algorithm 1 produces projective arrangements of a rooted tree $T^r$ u.a.r. is simple. The first call takes the root and its dependents and produces a uniformly random permutation with probability $1/(d(r) + 1)!$. Subsequent recursive calls (in Algorithm 2) produce the corresponding
Algorithm 1: Generating projective arrangements u.a.r.

Function \textsc{Random Projective Arrangement}(T') is
\begin{enumerate}
\item \textbf{Input}: \(T'\) a rooted tree.
\item \textbf{Output}: A projective arrangement \(\pi\) of \(T'\) chosen u.a.r.
\item \(\pi \leftarrow \text{empty } n\text{-vertex arrangement}\)
\end{enumerate}

// Algorithm 2
\begin{enumerate}
\item \textsc{Random Projective Arrangement Subtree}(\(T', r, 1, \pi\))
\item \textbf{return} \(\pi\)
\end{enumerate}

Algorithm 2: Generating projective arrangements u.a.r. of a subtree

Function \textsc{Random Projective Arrangement Subtree}(\(T', u, p, \pi\)) is
\begin{enumerate}
\item \textbf{Input}: \(T'\) a rooted tree, \(u\) any vertex of \(T'\), \(p\) the starting position to arrange the vertices of \(T'_u\), \(\pi\) partially-constructed without \(T'_u\).
\item \(\Phi_u \leftarrow \text{a random permutation of } \Gamma_r(u) \cup \{u\}\)
\item \textbf{for} \(v \in \Phi_u\) \textbf{do}
\item \quad \textbf{if} \(v = u\) \textbf{then}
\item \quad \quad \(\pi(v) \leftarrow p\)
\item \quad \quad \(p \leftarrow p + 1\)
\item \quad \textbf{else}
\item \quad \quad \textsc{Random Projective Arrangement Subtree}(\(T', v, p, \pi\))
\item \quad \quad \(p \leftarrow p + s_r(v)\)
\end{enumerate}

permutations each with its respective uniform probability, hence the probability of producing a particular permutation is the product of individual probabilities. Using Equation 6, we easily obtain that the probability of producing a certain projective arrangement is

\[ \prod_{u \in V} \frac{1}{(d_r(u) + 1)!} = \frac{1}{N_{pr}(T')} . \]

2.3.2 Generation of projective arrangements in past research

Algorithm 1 is equivalent to the “fully random” method used by Futrell et al. 2015 as witnessed by the implementation of their code available on Github,\(^5\) in particular in file cliqs/mindep.py\(^6\) (function \_randlin\_projective). Notice that Futrell et al. 2015 outline

\(^5\)https://github.com/Futrell/cliqs/tree/44bfcf2c42c848243c264722b5eccdfec0ede6a

\(^6\)https://github.com/Futrell/cliqs/blob/44bfcf2c42c848243c264722b5eccdfec0ede6a/cliqs/mindep.py
(though vaguely) that a projective arrangement is generated randomly by “Starting at the root node of a dependency tree, collect[ing] the head word and its dependents and order[ing] them randomly”.

Futrell et al. 2015 present their method to generate random projective arrangements as though it were the same as that by Gildea and Temperley 2007, 2010, who introduced a method to generate random linearizations of a tree which consists of “choosing a random branching direction for each dependent of each head, and – in the case of multiple dependents on the same side – randomly ordering them in relation to the head” (Gildea and Temperley 2010). However, Futrell et al. 2015 do not actually implement Gildea and Temperley’s method as witnessed by their code. Critically, Gildea and Temperley’s method does not produce uniformly random linearizations as we show with a counterexample.

Consider a star tree rooted at its hub. Let $X$ be a random variable for the position of the root in a random projective linear arrangement ($1 \leq X \leq n$). We have $P(X = x) = 1/n$ for all $x \in [1, n]$, therefore $X$ follows a uniform distribution and hence $E[X] = (n + 1)/2$ and $\mathbb{V}[X] = (n^2 - 1)/12$ (Mitzenmacher and Upfal 2017). Let $X'$ be a random variable for the position of the root according to Gildea and Temperley’s method. It is easy to see that $X' - 1$ follows a binomial distribution with parameters $n - 1$ and $1/2$. Namely, $P(X' - 1 = x) = \binom{n-1}{x} / 2^{n-1}$. We have that $E[X'] = 1 + E[X' - 1] = (n + 1)/2 = E[X]$, but $\mathbb{V}[X'] = \mathbb{V}[X' - 1] = (n - 1)/4$. Therefore, the variance in a truly uniformly random projective linear arrangement is $\Theta(n^2)$ while Gildea and Temperley’s method results in $\Theta(n)$, a much smaller dispersion. As $n \to \infty$, $X' - 1$ converges to a Gaussian distribution.

Gildea and Temperley’s method was introduced as a random baseline for the distance between syntactically-related words in languages and has been used with that purpose (Gildea and Temperley 2007, 2010; Temperley and Gildea 2018). Interestingly, the minimum baseline, namely, the minimum sum of dependency distances, results from placing the root at the center (Shiloach 1979; Chung 1984). The example above shows that Gildea and Temperley’s baseline tends to put the root at the center of the linear arrangement

---

**7** That is, as explained by Temperley and Gildea 2018, “choose a random assignment of each dependent to either the left or the right of its head.”
with higher probability than the truly uniform baseline. That behavior casts doubts on the power of that random baseline to investigate dependency distance minimization in languages since it tends to place the root at the center of the sentence, as expected from an optimal placement under projectivity (Gildea and Temperley 2007; Alemany-Puig et al. 2021) and does it with much lower dispersion around the center than in truly uniformly random linearizations.

2.3.3 Generating planar arrangements

Proposition 1 leads to a method to generate planar arrangements u.a.r. for any free tree $T$. The method we propose is detailed in Algorithm 3.

```
Function RANDOM_PLANAR_ARRANGEMENT(T) is
    Input: $T$ a free tree.
    Output: A planar arrangement $\pi$ of $T$ chosen u.a.r.
    $\pi \leftarrow$ empty $n$-vertex arrangement
    $u \leftarrow$ a vertex of $T$ chosen u.a.r.
    $\pi(u) \leftarrow 1$
    $\Phi_u \leftarrow$ a random permutation of $\Gamma(u)$
    $p \leftarrow 2$
    for $v \in \Phi_u$, do
        $p \leftarrow p + s_u(v)$
    return $\pi$
```

It is easy to see that Algorithm 3 has time complexity $O(n)$. Now we show that it generates planar arrangements uniformly at random. Firstly, choose a vertex, say $u \in V$, u.a.r., and place it at one of the arrangement’s ends, say, the leftmost position; this vertex acts as a root for $T$. Secondly, choose u.a.r. one of the $d(u)!$ permutations of the segments of the subtrees $T^u_v$ u.a.r. Lastly, recursively choose u.a.r. a projective linearization of every subtree $T^u_v$ for $v \in \Gamma(u)$ (Algorithm 2). These steps generate a planar arrangement u.a.r. since the probability of producing a certain planar arrangement following these steps is, then,

$$\frac{1}{n} \frac{1}{d(u)!} \prod_{v \in \Gamma(u)} \frac{1}{N_{pr}(T^u_v)} = \frac{1}{n} \frac{1}{d(u)!} \prod_{v \in V \setminus \{u\}} \frac{1}{d(v)!} = \frac{1}{N_{pl}(T)}.$$ 

The equalities follow from Proposition 1.
In this section we derive an arithmetic expression for $\mathbb{E}_{pl}[D(T)]$. First, we prove Theorem 1. To this aim, we define $\mathbb{E}_{pr}^\circ [\alpha_{uv} \mid r] = \mathbb{E}_{pr} [\alpha_{uv} \mid \pi(r) = 1]$ as the expected value of $\alpha_{uv}$ conditioned to the projective arrangements $\pi$ of $T^r$ such that $\pi(r) = 1$; we define $\mathbb{E}_{pr}^\circ [\beta_{uv} \mid r]$ likewise. The root is specified as a parameter of the expected value because we want to be able to use various roots. In the following proofs we rely heavily on Linearity of Expectation (Mitzenmacher and Upfal 2017, Theorem 2.1) and the Law of Total Expectation (Mitzenmacher and Upfal 2017, Lemma 2.5).

**Proof** [Proof of Theorem 1] We first prove Equation 4. By the Law of Total Expectation,

$$\mathbb{E}_{pl}[D(T)] = \sum_{u \in V} \mathbb{E}_{pl}[D(T) \mid \pi(u) = 1] P_{pl}(\pi(u) = 1).$$

Notice, quite simply, that

$$\mathbb{E}_{pl}[D(T) \mid \pi(u) = 1] = \mathbb{E}_{pr} [D(T^u) \mid \pi(u) = 1] = \mathbb{E}_{pr}^\circ [D(T^u)],$$

that is, the expected value of $D$ conditioned to planar arrangements of $T$ such that $u$ is fixed at the leftmost position, $\mathbb{E}_{pl}[D(T) \mid \pi(u) = 1]$, is equal to the expected value of $D$ conditioned to projective arrangements of $T^u$ such that vertex $u$ is fixed at the leftmost position, which is denoted as $\mathbb{E}_{pr}^\circ [D(T^u)]$. By noticing, given a fixed vertex $u$, that

$$P_{pl}(\pi(u) = 1) = \frac{1}{n},$$

which is the proportion of planar arrangements of $T$ in which $\pi(u) = 1$ (Proposition 1), Equation 4 follows immediately. Notice that Equation 4 expresses the expected value of $D$ conditioned to planar arrangements of a free tree $T$ as the average of each of the expected values of $D$ conditioned to projective arrangements of $T^u$ (for all $u \in V$) such that the root is fixed at the leftmost position.

Now we aim to write $\mathbb{E}_{pr}^\circ [D(T^u)]$ as a function of $\mathbb{E}_{pr} [D(T^u)]$. We start by decomposing $\mathbb{E}_{pr}^\circ [D(T^u)]$ into a summation of expected values of the individual edge lengths, and group the edges of every subtree $T^u_v$ of $T^u$ (where $uv$ is a (directed) edge of the tree) into one single expected value for each subtree and leave the edges incident to the root $u$ in the same summation as follows:

$$\mathbb{E}_{pr}^\circ [D(T^u)] = \sum_{vw \in \Gamma(u)} \left( \mathbb{E}_{pr}^\circ [\delta_{vw} \mid u] + \mathbb{E}_{pr} [D(T^u_v)] \right).$$
Now, it is important to notice that we did not write $\mathbb{E}^o_{pr}[D(T''_u)]$ in the summation above since the conditioning imposed by the operator $\diamond$ in $\mathbb{E}^o_{pr}[D(T'')]$ only applies to the root $u$. The root of the subtrees can be placed freely in the arrangement as long as the result is projective. Now we decompose all (directed) edges $uv$ of $T''$ in the first summation into anchor and coanchor, and we get

$$
\mathbb{E}^o_{pr}[D(T'')] = \sum_{v \in \Gamma(u)} \left( \mathbb{E}^o_{pr}[\alpha_{uv} + \beta_{uv} | u] + \mathbb{E}_{pr}[D(T''_v)] \right).
$$

Although the root $u$ is clear in this context, we have made it explicit in $\mathbb{E}_{pr}[\alpha_{uv} + \beta_{uv} | u]$ so as to be able to keep track of it in the following derivations. By linearity of expectation,

$$
\mathbb{E}_{pr}[\alpha_{uv} + \beta_{uv} | u] = \mathbb{E}_{pr}[\alpha_{uv} | u] + \mathbb{E}_{pr}[\beta_{uv} | u].
$$

Now, notice that the length of the anchor of any given directed edge $(u, v)$, where $u$ is the head and $v$ is the dependent, is invariant to the position of $u$, that is, it only changes if we change the position of $v$ within its interval. Therefore, fixing the head to the leftmost position of the arrangement (or any position outside the segment of $v$) does not affect the value of $\mathbb{E}_{pr}[\alpha_{uv} | u]$ and we simply have that $\mathbb{E}_{pr}[\alpha_{uv} | u] = \mathbb{E}_{pr}[\alpha_{uv} | u]$ and thus

$$
\mathbb{E}_{pr}[D(T'')] = \sum_{v \in \Gamma(u)} \left( \mathbb{E}_{pr}[\alpha_{uv} | u] + \mathbb{E}_{pr}[\beta_{uv} | u] \right)
+ \mathbb{E}_{pr}[D(T''_v)].
$$

The next step is to find the value of $\mathbb{E}_{pr}[\beta_{uv} | u]$. Notice now that the length of the coanchor of any directed edge $(u, v)$ is affected by the position of the head $u$ and, as such, $\mathbb{E}_{pr}[\beta_{uv} | u]$ need not be exactly equal to $\mathbb{E}_{pr}[\beta_{uv} | u]$. The derivation is found in Appendix 4.3 since it is merely an adaptation of the proof by Alemany-Puig and Ferrer-i-Cancho 2022, Lemma 1; it gives

$$
\mathbb{E}_{pr}[\beta_{uv} | u] = \frac{3}{2} \mathbb{E}_{pr}[\beta_{uv} | u].
$$
Thus,

\[
\mathbb{E}_\text{pr}^\circ [D(T^u)] = \sum_{v \in \Gamma(u)} \left( \mathbb{E}_\text{pr} [\alpha_{uv} | u] + \frac{3}{2} \mathbb{E}_\text{pr} [\beta_{uv} | u] + \mathbb{E}_\text{pr} [D(T^u_v)] \right)
\]

\[
= \sum_{v \in \Gamma(u)} \left( \mathbb{E}_\text{pr} [\delta_{uv} | u] + \mathbb{E}_\text{pr} [D(T^u_v)] + \frac{1}{2} \mathbb{E}_\text{pr} [\beta_{uv} | u] \right)
\]

\[
= \mathbb{E}_\text{pr} [D(T^u)] + \frac{1}{2} \sum_{v \in \Gamma(u)} \mathbb{E}_\text{pr} [\beta_{uv} | u].
\]

(8)

In the third equality we have used the identity by Alemany-Puig and Ferrer-i-Cancho 2022, Equation 28, which states that in a rooted tree \( T^r \)

\[
\mathbb{E}_\text{pr} [D(T^r)] = \sum_{v \in \Gamma(r)} \left( \mathbb{E}_\text{pr} [\delta_{rv}] + \mathbb{E}_\text{pr} [D(T^r_v)] \right).
\]

In this equation, we have not specified the expected values as being conditioned by the root \( r \) since this is clear from the context. Plugging Equation 8 into Equation 4 we get

\[
\mathbb{E}_\text{pl} [D(T)] = \frac{1}{2n} \sum_{u \in V} \sum_{v \in \Gamma(u)} \mathbb{E}_\text{pr} [\beta_{uv} | u] + \frac{1}{n} \sum_{u \in V} \mathbb{E}_\text{pr} [D(T^u)]
\]

(9)

We can use the following result by Alemany-Puig and Ferrer-i-Cancho 2022, Equation 16:

\[
\mathbb{E}_\text{pr} [\beta_{uv} | u] = \frac{s_u(u) - s_u(v) - 1}{3} = \frac{n - s_u(v) - 1}{3}
\]

to further simplify Equation 9 and, after proving that

\[
\sum_{v \in \Gamma(u)} \mathbb{E}_\text{pr} [\beta_{uv} | u] = \frac{s_u(u) - s_u(v) - 1}{3} = \frac{(n - 1)(d(u) - 1)}{3},
\]

\[
\sum_{u \in V} \frac{1}{3} (n - 1)(d(u) - 1) = \frac{(n - 1)(n - 2)}{3},
\]

[ 19 ]
we obtain
\begin{equation}
\frac{1}{2n} \sum_{u \in V} \sum_{v \in \Gamma(u)} \mathbb{E}_{pr}[\beta_{uv} \mid u] = \frac{(n-1)(n-2)}{6n}.
\end{equation}

Hence Equation 5.

For the sake of comprehensiveness, we also provide an arithmetic expression for the expected length of an edge $uv$ of a free tree in uniformly random planar arrangements. To this aim, we further define $\mathbb{E}_{pl}^\circ[\delta_{uv} \mid r] = \mathbb{E}_{pl}[\delta_{uv} \mid \pi(r) = 1]$ to be the expected value of the length of edge $uv \in E(T)$ when the vertex $r \in V(T)$ is fixed to the leftmost position in planar arrangements of $T$. Similarly, given a rooting of $T$ at $r$, let $\mathbb{E}_{pr}^\circ[\delta_{uv} \mid r] = \mathbb{E}_{pr}[\delta_{uv} \mid \pi(r) = 1]$ to be the expected value of the length of edge $uv \in E(T^r)$ when vertex $r$ acts as the root of the tree and it is fixed to the leftmost position in projective arrangements of $T^r$. The root vertex $r$ may be vertex $u$, vertex $v$, or neither. In the expected value $\mathbb{E}_{pr}^\circ[\delta_{uv} \mid r]$ we assume that the edge $uv$ is directed from $u$ to $v$ in accordance with the orientation defined by the root vertex $r$. Therefore, when $r$ is neither $u$ nor $v$, the vertex of edge $uv$ closest to $r$ is always vertex $u$, and the farthest is always vertex $v$.

**Lemma 2** Given a free tree $T = (V, E)$, for any $uv \in E$ it holds that
\begin{equation}
\mathbb{E}_{pl}^\circ[\delta_{uv}] = 1 + \frac{1}{n} \sum_{r \in V \setminus \{u, v\}} \mathbb{E}_{pr}[\delta_{uv} \mid r],
\end{equation}
where as per Alemany-Puig and Ferrer-i-Cancho 2022
\begin{equation}
\mathbb{E}_{pr}[\delta_{uv} \mid r] = \frac{2s_r(u) + s_r(v) + 1}{6}.
\end{equation}

**Proof** Following the characterization of planar arrangements described in Section 2.2, we have that $\mathbb{P}_{pl}(\pi(r) = 1) = 1/n$. Then applying the Law of Total Expectation
\begin{align}
\mathbb{E}_{pl}^\circ[\delta_{uv}] &= \sum_{r \in V} \mathbb{E}_{pl}^\circ[\delta_{uv} \mid \pi(r) = 1] \mathbb{P}_{pl}(\pi(r) = 1) \\
&= \frac{1}{n} \sum_{r \in V} \mathbb{E}_{pl}^\circ[\delta_{uv} \mid r].
\end{align}
(13)
Now we calculate $\mathbb{E}_{pl}^\circ[\delta_{uv} \mid r]$ by cases. When $r \notin \{u, v\}$,
\begin{equation}
\mathbb{E}_{pl}^\circ[\delta_{uv} \mid r] = \mathbb{E}_{pr}^\circ[\delta_{uv} \mid r] = \mathbb{E}_{pr}[\delta_{uv} \mid r].
\end{equation}
When \( r \in \{u, v\} \), by linearity of expectation,
\[
\mathbb{E}_{pl}^{o}[\delta_{uv} | r] = \mathbb{E}_{pr}^{o}[\delta_{uv} | r] \\
= \mathbb{E}_{pr}^{o}[\alpha_{uv} + \beta_{uv} | r] \\
= \mathbb{E}_{pr}^{o}[\alpha_{uv} | r] + \mathbb{E}_{pr}^{o}[\beta_{uv} | r].
\]

By denoting \( r \) the only vertex in \( \{u, v\} \setminus \{r\} \), then
\[
(15) \quad \mathbb{E}_{pr}^{o}[\alpha_{uv} | r] = \mathbb{E}_{pr}^{o}[\alpha_{uv} | r] = \frac{s_r(r) + 1}{2}.
\]

Equation 15 relies on the fact that in a rooted tree \( T^r \), the expected length of the anchor of an edge incident to the root, say \( rw \in E(T^r) \), is given by \( \mathbb{E}_{pr}^{o}[\alpha_{rw} | r] = (s_r(w) + 1)/2 \) (Alemany-Puig and Ferrer-i-Cancho 2022). An arithmetic expression for \( \mathbb{E}_{pr}^{o}[\beta_{uv} | r] \) can be found by modifying the proof of Alemany-Puig and Ferrer-i-Cancho 2022, Lemma 1. Then, as before, we get (see Appendix 4.3),
\[
(16) \quad \mathbb{E}_{pr}^{o}[\beta_{uv} | r] = \frac{3}{2} \mathbb{E}_{pr}^{o}[\beta_{uv} | r] = \frac{n - s_r(r) - 1}{2}.
\]

Therefore, by adding Equations 15 and 16 we obtain
\[
\mathbb{E}_{pl}^{o}[\delta_{uv} | r] = \mathbb{E}_{pr}^{o}[\alpha_{uv} | r] + \mathbb{E}_{pr}^{o}[\beta_{uv} | r] \\
= \frac{s_r(r) + 1}{2} + \frac{n - s_r(r) - 1}{2} \\
= \frac{n}{2}.
\]

Equation 11 follows immediately after inserting Equations 17 and 14 in Equation 13. \( \square \)

**APPLICATIONS**

A linear-time algorithm to compute \( \mathbb{E}_{pl}^{o}[D(T)] \)

Here we consider algorithms of increasing efficiency. First, since \( \mathbb{E}_{pr}^{o}[D(T^u)] \) can be calculated in \( O(n) \)-time for any \( n \)-vertex rooted tree \( T^u \) (Alemany-Puig and Ferrer-i-Cancho 2022, Theorem 1), the evaluation ‘as is’ of Equation 5 leads to a \( O(n^2) \)-time algorithm.

Second, we could calculate the value \( \mathbb{E}_{pr}^{o}[D(T^u)] \) for all \( u \in V \) in \( O(n) \)-time and \( O(n) \)-space with the following procedure:
1. Precompute $s_u(v)$ in $O(n)$-time (Alemany-Puig et al. 2022);
2. Choose an arbitrary vertex $w$;
3. Calculate $E_{pr}[D(T^w)]$ in $O(n)$-time (Alemany-Puig and Ferrer-i-Cancho 2022); and, finally,
4. Perform a Breadth First Search (BFS) traversal of $T$ starting at $w$.

In this traversal, when going from vertex $u$ to vertex $v$, the value of $E_{pr}[D(T^v)]$ is calculated applying the precomputed value of $E_{pr}[D(T^u)]$ to the following equation:

$$E_{pr}[D(T^u)] = E_{pr}[D(T^v)] + \Delta,$$

where $\Delta$ is equal to the difference $E_{pr}[D(T^u)] - E_{pr}[D(T^v)]$. We can obtain a formula for this difference by manipulating Equation 3. We get

$$\Delta = \frac{1}{6} [s_u(v)(2d(v) - 1) + 2n(d(u) - d(v))$$

$$- s_v(u)(2d(u) - 1)].$$

Notice that the value of $\Delta$ can be computed in constant time for any two vertices $u$ and $v$ (here we are interested in the value of $\Delta$ for pairs of adjacent vertices) and, crucially, without knowledge of either $E_{pr}[D(T^u)]$ or $E_{pr}[D(T^v)]$. That is, if the value of $E_{pr}[D(T^u)]$ is known then the value of $E_{pr}[D(T^v)]$ for any $v \in \Gamma(u)$ can be calculated in constant time as

$$E_{pr}[D(T^v)] = E_{pr}[D(T^u)] - \Delta.$$

Third, we propose an alternative that is also $O(n)$-time yet simpler and faster in practice, based on Proposition 2.

**PROPOSITION 2**

Given a free tree $T = (V, E)$,

$$E_{pl}[D(T)] = \frac{(n - 1)(3n^2 + 2n - 2)}{6n} - \frac{1}{6n} \sum_{v \in V} (2d(v) - 1) \sum_{u \in \Gamma(v)} s_v(u)^2.$$
**PROOF**  Here we simplify the summation in Equation 5, which becomes (as per Alemany-Puig and Ferrer-i-Cancho 2022)

\[
\frac{1}{n} \sum_{u \in V} \mathbb{E}_{pr} [D(T^u)] = \frac{1}{6n} (f(T) - n)
\]

with

\[
f(T) = \sum_{u \in V} \sum_{v \in V} s_u(v)(d_u(v) + 1).
\]

Now we simplify \(f(T)\) by first replacing the term \(d_u(v)\) by \(d(v)\) after the necessary transformations so that we can swap the order of the summations afterwards, that is,

\[
f(T) = \sum_{u \in V} \left( s_u(u)(2d_u(u) + 1) + \sum_{v \in V \setminus \{u\}} s_u(v)(2d_u(v) + 1) \right)
\]

\[
= \sum_{u \in V} n(2d(u) + 1) + \sum_{u \in V} \sum_{v \in V \setminus \{u\}} s_u(v)(2d(v) - 1)
\]

\[
= n(5n - 4) - \sum_{u \in V} s_u(u)(2d(u) - 1)
\]

\[
+ 2 \sum_{u \in V} \sum_{v \in V} s_u(v)d(v) - \sum_{u \in V} \sum_{v \in V} s_u(v)
\]

\[
(19) \quad = 2n^2 + g(T) - h(T)
\]

with

\[
(20) \quad g(T) = 2 \sum_{u \in V} \sum_{v \in V} s_u(v)d(v),
\]

\[
(21) \quad h(T) = \sum_{u \in V} \sum_{v \in V} s_u(v).
\]

In the preceding derivation, the second equality holds due to \(d_u(v) = d(v) - 1\) for \(v \neq u\); the third and fourth steps, we apply the Handshaking lemma. These lead to

\[
(22) \quad \frac{1}{n} \sum_{u \in V} \mathbb{E}_{pr} [D(T^u)] = \frac{1}{6n} (n(2n - 1) + g(T) - h(T)).
\]

---

8 The Handshaking lemma (Gunderson 2014) states that the sum of the degrees of all vertices of a graph equals twice the number of its edges.
Proof of Proposition 2. The value \( s_u(v) \) is the same for all vertices of \( T_w^v \) denoted as \( \{u_1, \ldots, u_k\} \) in the figure and the proof.

It remains to simplify Equations 20 and 21. We start by changing the order of the summations in Equation 20,

\[
g(T) = 2 \sum_{v \in V} \sum_{u \in V} s_u(v)d(v) = 2 \sum_{v \in V} d(v) \sum_{u \in V} s_u(v),
\]

and continue simplifying the inner summation. Consider a fixed \( v \in V \). We have that

\[
\sum_{u \in V} s_u(v) = n + \sum_{u \in V \setminus \{v\}} s_u(v) = n + \sum_{w \in \Gamma(v)} s_u(v)s_w(v).
\]

The summation (1) adds up the size of all subtrees \( T_w^v \) with respect to a ‘moving’ root \( w \). In the first equality we have simply taken out the case \( s_u(u) \). To understand the second equality, focus for now on a single subtree \( T_w^v \) such that \( wv \in E \). The summation (2) contains summands that correspond to all the vertices in \( T_w^v \), say vertices \( u_1, \ldots, u_k \) (assume, without loss of generality, that \( w = u_k \)). These summands are \( s_{u_1}(v), \ldots, s_{u_k}(v) \), which are all equal to \( s_w(v) \) (Figure 6). Moreover, there are \( s_w(v) \) vertices in \( T_w^v \) thus \( k = s_w(v) \), and this holds for all \( w \in \Gamma(v) \), hence the equality. Finally,

\[
\sum_{u \in V} s_u(v) = n + \sum_{w \in \Gamma(v)} s_w(v)s_w(v) = n^2 - \sum_{u \in \Gamma(v)} s_u(u)^2,
\]

thanks to the identity \( s_u(v) + s_v(u) = n \). Then,

\[
g(T) = 4n^2(n - 1) - 2 \sum_{v \in V} d(v) \sum_{u \in \Gamma(v)} s_u(u)^2.
\]

We use the result in Equation 23 to simplify Equation 21,

\[
h(T) = \sum_{v \in V} \sum_{u \in V} s_u(v) = n^3 - \sum_{v \in V} \sum_{u \in \Gamma(v)} s_u(u)^2.
\]
By combining Equations 24 and 25 into Equation 22 and, after some effort, we obtain

\[
\mathbb{E}_{pl}[D(T)] = \frac{(n-1)(n-2)}{6n} + \frac{1}{6n} \left( n(n-1)(3n+1) \right.
- \sum_{v\in V} (2d(v) - 1) \sum_{u\in \Gamma(v)} s_u(v)^2 ),
\]

which leads directly to Equation 18.

**Lemma 3** For any given free tree \( T \), Algorithm 4 calculates \( \mathbb{E}_{pl}[D(T)] \) in time and space \( O(n) \).

**Proof** The pseudocode to calculate \( \mathbb{E}_{pl}[D(T)] \) based on Proposition 2 is given in Algorithm 4. This algorithm first calculates \( s_u(v) \) for all edges \( uv \in E \), for the given tree \( T \) in \( O(n) \) time using the pseudocode by Alemany-Puig et al. 2022, Algorithm 2.1. Then it uses these values to calculate the sums of \( s_v(u)^2 \) for every vertex \( v \in V \). Such sums are then used to evaluate Equation 18 hence calculating \( \mathbb{E}_{pl}[D(T)] \) in time \( O(n) \).

---

**Algorithm 4:**

**Calculation of \( \mathbb{E}_{pl}[D(T)] \).
Cost \( O(n) \)-time, \( O(n) \)-space**

1. **Function** `COMPUTEEXPECTED_PLANAR(T)`
   - **Input:** \( T \) free tree.
   - **Output:** \( \mathbb{E}_{pl}[D(T)] \).
   - // Alemany-Puig et al. 2022, Algorithm 2.1
   2. \( S \leftarrow \{0\}^n \) // a vector of \( n \) zeroes.
   3. \( L \leftarrow \{0\}^n \) // a vector of \( n \) zeroes.
   4. **for** \((u,v,s_u(v))\) \( \in S \) **do** \( L[u] \leftarrow L[u] + s_u(v)^2 \)
   5. **return** \( ((n-1)(3n^2 + 2n - 2) - \sum_{u\in V} (d(u) - 1)L[u])/6n \)

---

A simple application

3.1.1

Let \( \mathbb{E}_{\geq 1}[D(T)] \) be the expected value of the sum of edge lengths conditioned to arrangements \( \pi \) such that \( C_\pi(T) \geq 1 \). That is, arrangements such that the number of edge crossings is at least 1. An immediate consequence of Lemma 3 is that \( \mathbb{E}_{\geq 1}[D(T)] \) can be computed easily as the following corollary states.
COROLLARY 3 For any free tree $T$, $\mathbb{E}_{\geq 1}[D(T)]$ can be computed in time and space $O(n)$ thanks to the fact that

$$\mathbb{E}_{\geq 1}[D(T)] = \frac{\mathbb{E}[D(T)] - \mathbb{E}_{pl}[D(T)] \mathbb{P}(C(T) = 0)}{\mathbb{P}(C(T) \geq 1)}$$

with $\mathbb{P}(C(T) \leq 0) = N_{pl}(T)/n!$ and $\mathbb{P}(C(T) \geq 1) = (n! - N_{pl}(T))/n!$.

PROOF Due to the Law of Total Expectation,

$$\mathbb{E}[D(T)] = \mathbb{E}_{pl}[D(T)] \mathbb{P}(C(T) = 0) + \mathbb{E}_{\geq 1}[D(T)] \mathbb{P}(C(T) \geq 1),$$

and hence Equation 26. $N_{pl}(T)$ can be computed in $O(n)$-time with Equation 6 and $\mathbb{E}_{pl}[D(T)]$ can be computed in time and space $O(n)$ (Lemma 3). Hence all the components in the right hand side of Equation 26 can be computed in time and space $O(n)$. □

3.2 Real syntactic dependency distances versus random baselines

Evidence that dependency distances are smaller than expected by chance can be obtained by random baselines of varying strength:

- None, $\mathbb{E}[D(T)]$, the expectation of $D(T)$ in unconstrained random linear arrangements (Ferrer-i-Cancho 2004).
- Planarity, $\mathbb{E}_{pl}[D(T)]$, the expectation of $D(T)$ in planar random linear arrangements (this article).
- Projectivity, $\mathbb{E}_{pr}[D(T')]$, the expectation of $D(T)$ in projective random linear arrangements (Alemany-Puig and Ferrer-i-Cancho 2022; Gildea and Temperley 2007).

This raises the questions of what would be the most appropriate baseline for research on dependency distance minimization. $\mathbb{E}_{pr}[D(T')]$ is by far the most widely used random baseline (Gildea and Temperley 2007; Liu 2008; Park and Levy 2009; Futrell et al. 2015).

Since planarity is a weaker condition than projectivity, $\mathbb{E}_{pl}[D(T)]$ implies a gain in coverage. Accordingly, there are more planar sentences than projective sentences in real texts (Havelka 2007; Gómez-Rodríguez and Gómez-Rodríguez-Rodríguez and g 2010, Table 1) and also in artificially-generated syntactic dependency structures (Gómez-Rodríguez et al. 2022, Figure 2).
However, surprisingly, $E_{\text{pl}}[D(T)]$ has never been used in research on the principle of dependency distance minimization. Here we aim to test the hypothesis that formal constraints mask the effects of the principle, a hypothesis that has already been confirmed on artificially-generated syntactic dependency structures (Gómez-Rodríguez et al. 2022).

Given the natural growth of dependency distance as sentence length increases (Ferrer-i-Cancho and Liu 2014; Ferrer-i-Cancho et al. 2022), we measure, for each sentence, the average dependency distance, namely $\langle d \rangle = D(T)/(n - 1)$ instead of the raw total sum $D(T)$ (a sentence of $n$ vertices has $n - 1$ syntactic dependencies when the structure is a tree). As, in addition to such a growth, the manifestation of the principle also depends on sentence length (the statistical bias towards shorter distances may disappear or become a bias in the opposite direction in short sentences; Ferrer-i-Cancho and Gómez-Rodríguez 2021; Ferrer-i-Cancho et al. 2022), we compare the actual dependency distances against the values predicted by the baselines in sentences of the same length.

Data and methods

We use the Parallel Universal Dependencies 2.6 collection (Zeman et al. 2020) for experimentation. To control for annotation style, we consider two versions of the collection: the collection with its original content-head annotation (PUD) and its transformation into Surface-Syntactic Universal Dependencies 2.6 (hereafter PSUD). By doing so, we cover two major competing annotation styles (Gerdes et al. 2018).

We borrow the preprocessing methods from previous research (Ferrer-i-Cancho et al. 2022). The main features of the processing are that nodes that are punctuation marks are removed and that the corpus remains fully parallel after the removal (Ferrer-i-Cancho et al. 2022). The preprocessed data is freely available as ancillary materials of the Linear Arrangement Library website.\(^9\)

With respect to previous accounts (Havelka 2007; Ferrer-i-Cancho et al. 2018; Gómez-Rodríguez and g 2010), our collections exhibit some remarkable statistical differences. First, the proportion of projective and planar sentences is higher in PUD, where the proportion of

\(^9\)https://cqllab.upc.edu/lal/universal-dependencies/
Table 2: Proportion (%) of projective and planar sentences in the PUD collection

<table>
<thead>
<tr>
<th>Language</th>
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<th>Planar</th>
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</thead>
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<td>96.2</td>
<td>96.3</td>
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<tr>
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<tr>
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<td>99.4</td>
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<tr>
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<td>86.7</td>
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<tr>
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<td>95.9</td>
</tr>
<tr>
<td>Finnish</td>
<td>96.4</td>
<td>96.7</td>
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<tr>
<td>French</td>
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<td>98.3</td>
</tr>
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<td>96.9</td>
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<th>Planar</th>
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<td>Japanese</td>
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<tr>
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<tr>
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</tr>
<tr>
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<td>97.2</td>
</tr>
<tr>
<td>Turkish</td>
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<td>94.1</td>
</tr>
</tbody>
</table>

Table 3: Proportion (%) of projective and planar sentences in the PSUD collection

<table>
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</tr>
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<td>Finnish</td>
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<table>
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<tr>
<th>Language</th>
<th>Projective</th>
<th>Planar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Italian</td>
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<td>94.6</td>
</tr>
<tr>
<td>Japanese</td>
<td>35.8</td>
<td>35.8</td>
</tr>
<tr>
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<td>75.8</td>
<td>77.1</td>
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<td>87.7</td>
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<tr>
<td>Spanish</td>
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<td>80.9</td>
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<tr>
<td>Swedish</td>
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<td>93.7</td>
</tr>
<tr>
<td>Thai</td>
<td>85.6</td>
<td>86.8</td>
</tr>
<tr>
<td>Turkish</td>
<td>87.6</td>
<td>88.3</td>
</tr>
</tbody>
</table>

non-projective or non-planar sentences does not exceed 10% in most cases (Tables 2 and 3). This proportion increases in PSUD; wherein, in two exceptional languages, Chinese and Hindi, it becomes larger than 50% (Table 3). Second, the difference between the proportion of non-projective and non-planar sentences is smaller than in previous reports (Gómez-Rodríguez and g 2010; Havelka 2007). Having said that, notice that our collections are fully parallel, and special care has been taken to keep annotation consistent across languages.

Given formal constraint ‘c’ (either ‘none’, ‘planarity’ (c = pl) or ‘projectivity’ (c = pr)) and sentence length n,

1. We calculate $D(T^r)$ for each $T^r$ and also calculate the expected
sum of edge lengths under ‘c’ different constraints (none, Equation 2; planarity, Equation 5; projectivity, Equation 3).

2. Then, for each sentence, we divide each by \( n - 1 \), to produce the mean length of its dependencies

\[
\langle d_c \rangle = \frac{D}{n - 1}
\]

and the expected mean of length of its dependencies under some constraint ‘c’

\[
\mathbb{E}[\langle d_c \rangle] = \frac{\mathbb{E}_c[D]}{n - 1}.
\]

3. Finally, we compute the average \( \langle d_c \rangle \) and the average \( \mathbb{E}[\langle d_c \rangle] \) over all sentences of length \( n \) satisfying constraint ‘c’.

Results

3.2.2

Figures 7 and 8 show the scaling of mean dependency distance as a function of sentence length in real sentences and in their corresponding random baselines. Concerning the random baselines (dashed lines), we find that the stronger the formal constraint on syntactic dependency structures, the lower the value of the random baseline. In contrast, the actual mean sentence length (solid lines) is practically the same independently of the formal constraint (none, planarity and projectivity). This is due to the fact the proportion of sentences that are lost by imposing some formal constraint is small in the PUD and PSUD collections, namely, the baselines \( \langle d \rangle \), \( \langle d_{pl} \rangle \) and \( \langle d_{pr} \rangle \) are extremely similar in value. The overwhelming majority of sentences are planar and the proportion of planar sentences that are not projective is really small (Table 2 and 3). Thus, selecting sentences satisfying a certain formal constraint has a negligible impact on the estimation of mean dependency distance.

Concerning the relationship between the actual mean dependency distance and the random baselines, we find that the average \( \langle d \rangle \) is below the average value of the random baselines for sufficiently large \( n \) in all languages. The only exception is Turkish, where the actual average \( \langle d \rangle \) is just slightly below the average of the projective baseline (Figures 7 and 8).
Figure 7:
The scaling of $\langle d \rangle$, the mean dependency distance of a sentence as a function of sentence length ($n$) for languages in the PUD collection for formal constraints of increasing strength: none (blue), planarity (green) and projectivity (red). Lines indicate the average value over all sentences of the same length. Solid lines are used for real sentences and dashed lines are used for the corresponding random baseline. Solid lines overlap so much that only one of them can be seen in most cases.
Figure 8: The scaling of \( <d> \), the mean dependency distance of a sentence as a function of sentence length \( n \) for languages in the PSUD collection for formal constraints of increasing strength. Format is the same as in Figure 7. Again, solid lines overlap so much that only one of them can be seen in most cases.
These findings are consistent between PUD and PSUD, in spite of their differences in proportions of projective and planar sentences commented above.

4 CONCLUSIONS AND FUTURE WORK

4.1 Theory

In Section 2.2, we have characterized planar arrangements of a given free tree $T$ using the concept of segment (Alemany-Puig and Ferrer-i-Cancho 2022). Employing said characterization, we have shown that the number of planar arrangements of a free tree depends on its degree sequence (Proposition 1), similar to the manner in which projective arrangements of a rooted tree do (Alemany-Puig and Ferrer-i-Cancho 2022). Moreover, we have given a procedure to generate u.a.r. planar arrangements of a given free tree in Section 2.3 (Algorithm 3) which can be easily adapted to generate such arrangements exhaustively. Notably, our algorithm to generate planar arrangements is based on the generation of projective arrangements of a rooted subtree. For the sake of completeness, we have detailed a procedure to generate u.a.r. projective arrangements of a given rooted tree (Algorithm 1).

4.2 Applications

Having identified the underlying structure of planar arrangements, we have derived an arithmetic expression, in Section 2.4, for $E_{pl}[D(T)]$ (Theorem 1). We have also devised a $O(n)$-time algorithm to calculate this value (Proposition 1, Algorithm 4).

In Section 3, we have applied the theory developed up until that point to investigate the effect of formal constraints of increasing strength (none, planarity, projectivity) in a parallel collection and reported two main findings. First, the average dependency distance in real sentences remains practically the same while the strength of the formal constraint increases. We believe that this result stems from the high proportion of planar sentences (and the very low proportion of planar sentences that are not projective) of the PUD collection. Higher
proportions of non-planar sentences have been reported in other collections (Gómez-Rodríguez and Ferrer-i-Cancho 2017). Second, the tendency of the random baseline to have a smaller value in stronger formal constraints indicates that the strength of the dependency distance minimization effect depends on the choice of the formal constraint for the random baseline. As these formal constraints may be a side effect of dependency distance minimization (Ferrer-i-Cancho 2006; Gómez-Rodríguez and Ferrer-i-Cancho 2017; Gómez-Rodríguez et al. 2022; Yadav et al. 2022), this phenomenon suggests that

1. Formal constraints absorb the dependency distance effect.
2. A fairer evaluation of the actual degree of optimization of dependency distances or a more accurate measurement of the power of the effect of dependency distance minimization requires considering not only the magnitude of the effect with respect to some random baseline but also the formal constraint, as the latter may hide part of the dependency distance minimization effect.

In past research on syntactic dependency distance minimization, \( \mathbb{E}_{E\text{pr}}[D(T')] \) has been the most widely used random baseline (Gildea and Temperley 2007; Liu 2008; Park and Levy 2009; Futrell et al. 2015). However, projectivity has a lower coverage than planarity in real sentences (Havelka 2007; Gómez-Rodríguez and G 2010). Projectivity is at risk of underestimating the strength of the dependency distance minimization principle (Ferrer-i-Cancho 2004) because of the significant reduction in the value of the random baseline (Figures 7 and 8) or the reduction of the actual dependency distances (Gómez-Rodríguez et al. 2022, Figure 2) that it introduces. Thanks to the research in this article, we have paved the way for replicating past research replacing \( \mathbb{E}_{E\text{pr}}[D(T')] \) with \( \mathbb{E}_{E\text{pl}}[D(T)] \).

### Future work

Planarity is a relaxation of projectivity but future work should address the problem of the expected value of \( D(T) \) in classes of formal constraints with even more coverage (Ferrer-i-Cancho et al. 2018). A promising step is the investigation of \( \mathbb{E}_{E\text{pl}}[D(T)] \), the expected value of \( D(T) \) conditioned to arrangements \( \pi \) such that \( C_{\pi}(T) \leq k \), that is,
in arrangements such that the number of edge crossings is at most \( k \). Notice that \( \mathbb{E}_{\leq 0}[D(T)] = \mathbb{E}_{pl}[D(T)] \). In real languages, the average number of crossings ranges between 0.40 and 0.62 (Ferrer-i-Cancho et al. 2018), suggesting that \( \mathbb{E}_{\leq k}[D(T)] \) with \( k = 1 \) or a small \( k \) would suffice.

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APPENDIX

DERIVATION OF \( E_{pr}^\alpha[\beta_{uv} | u] \)

Here we derive the expected length of the coanchor of a (directed) edge \( uv \in E(T^u) \) in uniformly random projective arrangements of \( T^u \) conditioned to \( \pi(u) = 1 \). Following Alemany-Puig and Ferrer-i-Cancho (2022), we decompose the length of the coanchor of the (directed) edge \( uv \), \( \beta_{uv} \), as the sum of the lengths of the segments in-between \( u \) and \( v \) (Figure 4). Here we use \( k_{uv} \) to denote the number of segments in-between \( u \) and \( v \), and \( \varphi_{uv}^{(i)} \) to denote the size of the \( i \)th segment, yielding (Alemany-Puig and Ferrer-i-Cancho 2022),

\[
\beta_{uv} = \sum_{i=1}^{k_{uv}} \varphi_{uv}^{(i)}.\]
By the Law of Total Expectation, we have that

\[
E^\circ_{pr} [\beta_{uv} | u] = \sum_{k=1}^{d(u)-1} E^\circ_{pr} [\beta_{uv} | u, k_{uv} = k] P^\circ_{pr} (k_{uv} = k | u),
\]

where \( E^\circ_{pr} [\beta_{uv} | u, k_{uv} = k] \) is the expectation of \( \beta_{uv} \) given that \( u \) is the root of the tree (fixed at the leftmost position), and that \( u \) and \( v \) are separated by \( k \) segments, and \( P^\circ_{pr} (k_{uv} = k | u) \) is the probability that \( u \) and \( v \) are separated by \( k \) intermediate segments, both in uniformly random projective arrangements \( \pi \) conditioned to \( \pi(u) = 1 \), both conditioned to the root of the tree being vertex \( u \). On the one hand,

\[
E^\circ_{pr} [\beta_{uv} | u, k_{uv} = k] = E^\circ_{pr} \left[ \sum_{i=1}^{k} \varphi_{uv}^{(i)} | u \right] = \frac{n - s_u(v) - 1}{d(u) - 1} k.
\]

Notice that this is the same result as that obtained in Alemany-Puig and Ferrer-i-Cancho 2022. Lastly, the proportion of arrangements in which the segment of \( v \) is at position \( k_{uv} + 1 \) equals \( (d(u) - 1)! \), therefore,

\[
P^\circ_{pr} (k_{uv} = k | u) = \frac{(d(u) - 1)! \prod_{v \in \Gamma(u)} N_{pr}(T^u)}{d(u)! \prod_{v \in \Gamma(u)} N_{pr}(T^u)} = \frac{1}{d(u)}.
\]

Recalling that (Alemany-Puig and Ferrer-i-Cancho 2022)

\[
E_{pr} [\beta_{uv} | u] = \frac{s_u(u) - s_u(v) - 1}{3},
\]

and plugging the results of Equations 29 and 30 into Equation 28, we get

\[
E^\circ_{pr} [\beta_{uv} | u] = \frac{n - s_u(v) - 1}{d(u) - 1} \frac{1}{d(u)} \sum_{k=1}^{d(u)-1} k
= \frac{s_u(u) - s_u(v) - 1}{2}
= \frac{3}{2} E_{pr} [\beta_{uv} | u].
\]
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Edge lengths in random planar linearizations


Alemany-Puig, Ferrer-i-Cancho

ECKHOFF, Marhaba ELI, Ali ELKAHKY, Binyam EPHREM, Olga ERINA, Tomáš ERJAVEC, Aline ETIENNE, Wograiné EVELYN, Richárd FARKAS, Hector FERNANDEZ ALCALDE, Jennifer FOSTER, Cláudia FREITAS, Kazunori FUJITA, Katarína GAJDOŠOVÁ, Daniel GALBRAITH, Marcos GARCIA, Moa GÄRDENFORS, Sebastian GARZA, Kim GERDES, Filip GINTER, Iakes GOENAGA, Koldo GOJENOLA, Memduh GÖKIRMAK, Yoav GOLDBERG, Xavier GÓMEZ GUINOVART, Berta GONZÁLEZ SAAVEDRA, Bernadeta GRICIÜTĖ, Matías GRIONI, Loïc GROBOL, Normunds GRŮZĪTIS, Bruno GUILLAUME, Céline GUILLOT-BARBANCE, Tunga GÜNGÖR, Nizar HABASH, Jan HAJIČ, Jan HAJIČ JR., Mika HÄMALÄINEN, Linh HÀ MỸ, Na-Rae HAN, Kim HARRIS, Dag HAVG, Johannes HEINECKE, Oliver HELLWIG, Felix HENNIG, Barbora HLADKÁ, Jaroslava HLAVÁČOVÁ, Florinel HOCIUNG, Petter HOHLE, Jena HWANG, Takumi IKEDA, Radu ION, Elena IRIMIA, Olájide ISHOLA, Tomáš JELÍNEK, Anders JOHANNSEN, Hildur JÓNSDÓTTIR, Fredrik JØRGENSEN, Markus JUUTINEN, Hüner KAŞıKARA, André KAASEN, Nadezha KABAЕVA, Sylvain KAHANE, Hiroshi KANAYAMA, Jenna KANERVA, Boris KATZ, Tolga KAYADELEN, Jessica KENNEY, Václava KETTNEROVÁ, Jesse KIRCHNER, Elena KLEMENTIEVA, Arne KÖHN, Abdullatif KÖKSAL, Kamil KOPACEWICZ, Timo KORKIANGAS, Natalia KOTSYBA, Jolanta KOVALEVSKAITĖ, Simon KREK, Sookyoung KWAK, Veronika LAIPPALA, Lorenzo LAMBERTINO, Lucia LAM, Tatiana LANDO, Septina Dian LARASATI, Alexei LAVRENTIEV, John LEE, Phùtnơng LÊ HỌNG, Alessandro LENCI, Saran LERTPRADIT, Herman LEUNG, Maria LEVINA, Cheuk Ying Li, Josie Li, Keying Li, KyungTae LIM, Yuan Li, Nikola LJUBEŠIĆ, Olga LOGINOVA, Olga LYASHEVSKAYA, Teresa LYNN, Vivien MACKETANZ, Aibek MAKAZHANOV, Michael MANDL, Christopher MANNING, Ruli MANURUNG, Cátălina MĂRĂNDUC, David MAREČEK, Katrin MARHEINECKE, Héctor MARTÍNEZ ALONSO, André MARTINS, Jan MAŠEK, Hiroshi MATSUDA, Yuji MATSUMOTO, Ryan MCDONALD, Sarah MCGUINNESS, Gustavo MENDONÇA, Niko MIEKKA, Margarita MISIRPASHAYEVA, Anna MISSILÄ, Cátălina MITITELU, Maria MITROFAN, Yusuke MIYAO, Simonetta MONTEMAGNI, Amir MORE, Laura MORENO ROMERO, Keiko Sophie MORI, Tomohiko MORIZUKA, Shinsuke MORI, Shigeki MORO, Bjartur MORTENSEN, Bohdan MOSKALEVSKYI, Kadri MUISCHNEK, Robert MUNRO, Yugo MURAWAKI, Kaili MÜÜRISIP, Pinkey NAINWANI, Juan Ignacio NAVARRO HORÑIACEK, Anna NEDOLUZHKO, Gunta NEŠPORE-BĒRZKALNE, Lưọtọng NGUYỄN THỊ, Huỳện NGUYỄN THỊ MINH, Yoshihiro NIKAIKO, Vitaly NIKOLAEV, Rattima NITTISAROJ, Hanna NURMI, Sīna OJALU, Atul K. OJHA, Adédayọ OLÔKUN, Mai OMURA, Emeka ONWUEGBUZIA, Petya OSENKOVA, Robert ÖSTLING, Lilja ÖVRELID, Şaziye BETÜL ÖZATES, ArzucaN ÖZGÜR, Balkız ÖZTÜRK BAŞARAN, Niko PARTANEN, Elena PASCUAL, Marco PASSAROTTI, Agnieszka PATEJUK, Guilherme PAULINO-PASSOS, Angelika PELJAK-LAPIŃSKA, Siyao PENG, Cenel-Augusto Perez, Guy PERRIER, Daria PETROVA, Slav PETROV, Jason PHELAN, Jussi PIITULAINEN, Tommi A
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